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Finite Unification and Top Quark Mass †

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Abstract

In unified gauge theories there exist renormalization group invariant relations among gauge and Yukawa couplings that are compatible with perturbative renormalizability, which could be considered as a Gauge-Yukawa Unification. Such relations are even necessary to ensure all-loop finiteness in Finite Unified Theories, which have vanishing β-functions beyond the unification point. We elucidate this alternative way of unification, and then present its phenomenological consequences in $SU(5)$-based models.

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1 Introduction

The original unification philosophy [1, 2] relates the gauge and separately the Yukawa couplings. A logical extension is to relate the couplings of the two sectors; Gauge-Yukawa Unification (GYU). Within the assumption that all the particles appearing in a theory are elementary, the theories based on extended supersymmetries [3] and string theories [4] are well-known possibilities for GYU. Unfortunately, these theories seem to introduce more serious and difficult phenomenological problems to be solved than those of the standard model.

There exists an alternative way to unify couplings which is based on the fact that within the framework of renormalizable field theory, one can find renormalization group invariant (RGI) relations among parameters and improve in this way the calculability and predictive power of a given theory [5]-[8]. We would like to briefly outline this idea below. Any RGI relation among couplings (which does not depend on the renormalization scale μ explicitly) can be expressed, in the implicit form $\Phi(g_1, \cdots, g_A) \;=\;$ const., which has to satisfy the partial differential equation (PDE)

$$\mu \frac{d\Phi}{d\mu} \;=\; \vec{\nabla}\cdot\vec{\beta} \;=\; \sum_{a=1}^{A} \beta_a \frac{\partial \Phi}{\partial g_a} \;=\; 0 \;, \tag{1}$$

where β_a is the β-function of g_a. This PDE is equivalent to the set to ordinary differential equations, the so-called reduction equations (REs)[5],

$$\beta_g \frac{dg_a}{dg} \;=\; \beta_a \;,\; a = 1, \cdots, A \;, \tag{2}$$

where g and β_g are the primary coupling and its β-function, and a does not include it. Since maximally $(A-1)$ independent RGI "constraints" in the A-dimensional space of couplings can be imposed by Φ_a's, one could in principle express all the couplings in terms of a single coupling g. The strongest requirement is to demand power series solutions to the REs,

$$g_a \;=\; \sum_{n=0} \rho_a^{(n+1)}\, g^{2n+1} \;, \tag{3}$$

which formally preserve perturbative renormalizability. Remarkably, the uniqueness of such power series solutions can be decided already at the one-loop level [5]. To illustrate

this, let us assume that the β-functions have the form

$$\begin{aligned}\beta_a &= \frac{1}{16\pi^2}[\sum_{b,c,d\neq g}\beta_a^{(1)\,bcd} g_b g_c g_d + \sum_{b\neq g}\beta_a^{(1)\,b} g_b g^2] + \cdots \ ,\\ \beta_g &= \frac{1}{16\pi^2}\beta_g^{(1)} g^3 + \cdots \ ,\end{aligned} \tag{4}$$

where $\cdots$ stands for higher order terms, and $\beta_a^{(1)\,bcd}$'s are symmetric in b, c, d. We then assume that the $\rho_a^{(n)}$'s with $n \leq r$ have been uniquely determined. To obtain $\rho_a^{(r+1)}$'s, we insert the power series (3) into the REs (2) and collect terms of $O(g^{2r+3})$ and find

$$\sum_{d\neq g} M(r)_a^d\,\rho_d^{(r+1)} \quad = \quad \text{lower order quantities} \ ,$$

where the r.h.s. is known by assumption, and

$$M(r)_a^d = 3\sum_{b,c\neq g}\beta_a^{(1)\,bcd}\,\rho_b^{(1)}\,\rho_c^{(1)} + \beta_a^{(1)\,d} - (2r+1)\,\beta_g^{(1)}\,\delta_a^d \ , \tag{5}$$

$$0 = \sum_{b,c,d\neq g}\beta_a^{(1)\,bcd}\,\rho_b^{(1)}\,\rho_c^{(1)}\,\rho_d^{(1)} + \sum_{d\neq g}\beta_a^{(1)\,d}\,\rho_d^{(1)} - \beta_g^{(1)}\,\rho_a^{(1)} \ . \tag{6}$$

Therefore, the $\rho_a^{(n)}$'s for all $n > 1$ for a given set of $\rho_a^{(1)}$'s can be uniquely determined if $\det M(n)_a^d \neq 0$ for all $n \geq 0$.

The possibility of coupling unification described above is without any doubt attractive because the "completely reduced" theory contains only one independent coupling, but it can be unrealistic. Therefore, one often would like to impose fewer RGI constraints, and this is the idea of partial reduction [6]. Among the existing possibilities in the framework of susy $SU(5)$ GUTs, there are two models that are singled out by being strongly motivated [7, 8]. The first is the $SU(5)$-Finite Unified Theory (FUT) [7]. In this theory, there exist RGI relations among gauge and Yukawa couplings that yield the vanishing of all β-functions to all orders in perturbation theory [9]. (It has been recently found that the quantum corrections to the cosmological constant in a finite theory is weakened [10].) The second is the minimal $SU(5)$ susy model which can be successfully partially-reduced [8]. This model is attractive because of its simplicity. In the following, we will give more emphasis in discussing the $SU(5)$-FUT and then we compare the predictions of the two models.

2 $N=1$ Finiteness

Let us consider a chiral, but anomaly free, globally supersymmetric gauge theory based on a simple group G with the gauge coupling g. The superpotential of the theory is given by

$$W = \sum_{i,j} \frac{1}{2} m_{ij}\, \phi^i\, \phi^j + \frac{1}{6} \sum_{i,j,k} \lambda_{ijk}\, \phi^i\, \phi^j\, \phi^k \ , \tag{7}$$

where the matter chiral superfield ϕ^i belongs to an irreducible representation of G. The non-renormalization theorem ensures that there are no extra mass and cubic-interaction-term renormalizations, implying that the β-functions of λ_{ijk} can be expressed as linear combinations of the anomalous dimension matrix γ_{ij} of ϕ^i. Therefore, all the one-loop β-functions of the theory vanish if

$$\beta_g^{(1)} = 0 \text{ and } \gamma_{ij}^{(1)} = 0 \tag{8}$$

are satisfied, where $\beta_g^{(1)}$ and $\gamma_{ij}^{(1)}$ are the one-loop coefficients of β_g and γ_{ij}, respectively. A very interesting result is that these conditions (8) are necessary and sufficient for finiteness at the two-loop level [11].

A natural question is what happens in higher loops. Since the finiteness conditions impose relations among couplings, they have to be consistent with the REs (1). (This should be so even for the one-loop finiteness.) Interestingly, there exists a powerful theorem [9] which provides the necessary and sufficient conditions for finiteness to all loops. The theorem makes heavy use of the non-renormalization property of the supercurrent anomaly [12]. In fact, the finiteness theorem can be formulated in terms of one-loop quantities, and it states that for susy gauge theories we are considering here, the necessary and sufficient conditions for β_g and β_{ijk} to vanish to all orders are [9]:

(a) The validity of the one-loop finiteness conditions, i.e., eq. (8) is satisfied.

(b) The REs (2) admit a unique power series solution, i.e., the corresponding matrix M defined in eq. (5) with $\beta_g^{(1)} = 0$ has to be non-singular.

The latter condition is equivalent to the requirement that the one-loop solutions $\rho_a^{(1)}$'s are isolated and non-degenerate. Then each of these solutions can be extended, by a recursion

formula, to a formal power series in g giving a theory which depends on a single coupling g, and has β-functions vanishing to all orders.

3 Finite Unified Models based on $SU(5)$

From the classification of theories with $\beta_g^{(1)} = 0$ [13], one can see that using $SU(5)$ as gauge group there exist only two candidate models which can accommodate three fermion generations. These models contain the chiral supermutiplets $\mathbf{5}$, $\overline{\mathbf{5}}$, $\mathbf{10}$, $\overline{\mathbf{5}}$, $\mathbf{24}$ with the multiplicities $(6, 9, 4, 1, 0)$ and $(4, 7, 3, 0, 1)$, respectively. Only the second one contains a $\mathbf{24}$-plet which can be used for spontaneous symmetry breaking (SSB) of $SU(5)$ down to $SU(3) \times SU(2) \times U(1)$. (For the first model one has to incorporate another way, such as the Wilson flux breaking to achieve the desired SSB of $SU(5)$.) Here we would like to concentrate only on the second model.

The most general $SU(5)$ invariant, cubic superpotential of the (second) model is:

$$\begin{aligned} W &= H_a \, [\, f_{ab} \, \overline{H}_b \mathbf{24} + h_{ia} \, \overline{\mathbf{5}}_i \mathbf{24} + \overline{g}_{ija} \, \mathbf{10}_i \overline{\mathbf{5}}_j] + p \, (\mathbf{24})^3 \\ &+ \frac{1}{2} \mathbf{10}_i \, [\, g_{ija} \, \mathbf{10}_j H_a + \hat{g}_{iab} \, \overline{H}_a \overline{H}_b + g'_{ijk} \, \overline{\mathbf{5}}_j \overline{\mathbf{5}}_k \,] \; , \end{aligned} \tag{9}$$

where $i, j, k = 1, 2, 3$ and $a, b = 1, \cdots, 4$, and we sum over all indices in W (the $SU(5)$ indices are suppressed). The $\mathbf{10}_i$'s and $\overline{\mathbf{5}}_i$'s are the usual three generations, and the four $(\mathbf{5} + \overline{\mathbf{5}})$ Higgses are denoted by H_a , $\overline{H}_a$.

Given the superpotential, the $\gamma^{(1)}$'s can be easily computed ($\beta_g^{(1)}$ vanishes of course). To ensure finiteness of the model to all orders, we have to find $\rho^{(1)}$'s that are isolated and non-degenerate solutions of eq. (6) and are consistent with the vanishing $\gamma^{(1)}$'s. In most of the previous studies of the present model [14], however, no attempt was made to find isolated and non-degenerate solutions, but rather the opposite. They have used the freedom offered by the degeneracy in order to make specific ansätze that could lead to phenomenologically acceptable predictions (see also [15]). Here we concentrate on finding an isolated and non-degenerate solution that is phenomenologically interesting. As a first approximation to the Yukawa matrices, a diagonal solution, that is, without intergenerational mixing, may be considered. It has turned out that this can be achieved

by imposing the $Z_7 \times Z_3$ discrete symmetry and a multiplicative Q-parity on W, and that, in order to respect these symmetries, only g_{iii} , $\overline{g}_{iii}$, f_{ii} and p are allowed to be non-vanishing. Moreover, we have found that under this situation there exists a unique reduction solution that satisfies the finiteness conditions (a) and (b) [7]:

$$\begin{aligned} g_{iii}^2 &= \frac{8}{5} g^2 + O(g^4) \ , \ \overline{g}_{iii}^2 = \frac{6}{5} g^2 + O(g^4) \ , \ f_{ii} = 0 \ , \\ f_{44}^2 &= g^2 + O(g^4) \ , \ p^2 = \frac{15}{7} g^2 + O(g^4) \ , \end{aligned} \tag{10}$$

where $i = 1, 2, 3$, and the $O(g^4)$ terms are power series in g that can be uniquely computed to any finite order if the β-functions of the unreduced model are known to the corresponding order. The reduced model in which gauge and Yukawa couplings are unified has the β-functions that identically vanish to that order.

4 Phenomenological Consequences

In the above model, we found a diagonal solution for the Yukawa couplings, with each family coupled to a different Higgs. However, we may use the fact that mass terms do not influence the β-functions in a certain class of renormalization schemes, and introduce appropriate mass terms that permit us to perform a rotation in the Higgs sector such that only one pair of Higgs doublets, coupled to the third family, remains light and acquires a non-vanishing v.e.v. Note that the effective coupling of the Higgs doublets to the first family after the rotation is very small avoiding in this way a potential problem with the proton lifetime [16]. Thus, effectively, we have at low energies the minimal susy standard model with only one pair of Higgs doublets. Adding soft breaking terms (which are supposed not to influence the β-functions beyond M_{GUT}), we can obtain susy breaking. The conditions on the soft breaking terms to reserve one-loop finiteness are given in [17]. Recently, the same problem at the two-loop level has been addressed [18]. It is an open problem whether there exists a suitable set of conditions on the soft terms for all-loop finiteness. Since the $SU(5)$ symmetry is spontaneously broken below M_{GUT}, the finiteness conditions obviously do not restrict the renormalization property at low energies, and all it remains is a boundary condition on the gauge and Yukawa couplings; these couplings

at low energies have to be so chosen that they satisfy (10) at M_{GUT}. So we examine the evolution of the gauge couplings according to their renormalization group equations at two-loops. Representative results are summarized in table 1. (To simplify our numerical analysis we assume a unique threshold M_S for all the superpartners.)

M_S [TeV]	$\alpha_S(M_Z)$	$\tan\beta$	m_b [GeV]	$m_t[GeV]$
1.0	0.117	54.1	5.13	185
0.5	0.121	53.5	5.27	186
0.2	0.121	54.1	5.14	185

Table 1. The predictions for different M_S, where we have used: $m_\tau = 1.78$ GeV, $\alpha_{em}^{-1}(M_Z) = 127.9$ and $\sin\theta_W(M_Z) = 0.232$.

All the quantities except M_S in table 1 are predicted. The dimensionless parameters (except $\tan\beta$) are defined in the $\overline{\mathrm{MS}}$ scheme, and the masses are pole masses. We see from table 1 that the low energy predictions are relatively stable against the change ofM_S and m_t agrees with the CDF result [19].

To compare the predictions above with those of the partially-reduced, minimal $SU(5)$ GUT [8], we present its predictions in table 2.

M_S [TeV]	$\alpha_S(M_Z)$	$\tan\beta$	m_b [GeV]	$m_t[GeV]$
1.0	0.118	47.4	5.36	180
0.5	0.120	47.6	5.42	180
0.2	0.124	47.4	5.55	182

Table 2. The predictions of the partially-reduced, minimal $SU(5)$ GUT for the same low-energy inputs.

We see from table 1 and 2 that the predictions of the partially-reduced, minimal $SU(5)$ GUT do not differ very much from these of the $SU(5)$-FUT model.

We would like to stress that both models have the strongest predictive power as compared with any other known GUTs as it was promised.

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References

[1] H. Georgi and S.L. Glashow, Phys. Rev. Lett. **32** (1974) 438

[2] H. Fritzsch and P. Minkowski, Ann. Phys. **93** (1975) 193; H. Georgi, in *Particles and Fields–1974*, ed. C.E. Carlson, American Institute of Physics, New York

[3] P. Fayet, Nucl. Phys. **B149** (1979) 134; F. del Aguila, M. Dugan, B. Grinstein, L. Hall, G.G. Ross and P. West, Nucl. Phys. **B250** (1985) 225

[4] M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory Vol. I,II*, Cambridge UP, Cambridge, 1987; D. Lüst and S. Theisen, *Lectures on String Theory*, Springer Notes in Physics, Vol. 346, Heidelberg, 1989

[5] W. Zimmermann, Commun. Math. Phys. **97** (1985) 211; R. Oehme and W. Zimmermann, Commun. Math. Phys. **97** (1985) 569; R. Oehme, Prog. Theor. Phys. Suppl. **86** (1986) 215

[6] J. Kubo, K. Sibold and W. Zimmermann, Nucl. Phys. **B259** (1985) 331; Phys. Lett. **B200** (1989) 185; J. Kubo, Phys. Lett. **B262** (1991) 472

[7] D. Kapetanakis, M. Mondragón and G. Zoupanos, Z. Phys. **C60** (1993) 181; M. Mondragón and G. Zoupanos, CERN preprint, CERN-TH.7098/93

[8] J. Kubo, M. Mondragón and G. Zoupanos, to appear in Nucl. Phys. **B**

[9] C. Lucchesi, O. Piguet and K. Sibold, Helv. Phys. Acta. **61** (1988) 321

[10] E.Elizalde and S. Odintsov, Phys. Lett. **B333** (1994) 331; I.L. Shapiro, Phys. Lett. **B329** (1994) 181

[11] A.J. Parkes and P.C. West, Phys. Lett. **B138** (1984) 99; Nucl. Phys. **B256** (1985) 340; D.R.T. Jones and A.J. Parkes, Phys. Lett. **B160** (1985) 267; D.R.T. Jones and L. Mezinescu, Phys. Lett. **B136** (1984) 242; B138 (1984) 293; A.J. Parkes, Phys. Lett. **B156** (1985) 73

[12] O. Piguet and K. Sibold, Int. Journ. Mod. Phys. **A1** (1986) 913

[13] S. Hamidi, J. Patera and J.H. Schwarz, Phys. Lett. **B141** (1984) 349; X.D. Jiang and X.J. Zhou, Phys. Lett. **B197** (1987) 156; **B216** (1985) 160

[14] S. Hamidi and J.H. Schwarz, Phys. Lett. **B147** (1984) 301; D.R.T. Jones and S. Raby, Phys. Lett. **B143** (1984) 137; J.E. Bjorkman, D.R.T. Jones and S. Raby, Nucl. Phys. **B259** (1985) 503, J. León et al, Phys. Lett. **B156** (1985) 66

[15] A.V. Ermushev, D.I. Kazakov and O.V. Tarasov, Nucl. Phys. **B281** (1987) 72; D.I. Kazakov, Mod. Phys. Lett. **A2** (1987) 663; Phys. Lett. **B179** (1986) 352

[16] N. Deshpande, Xiao-Gang, He and E. Keith, Phys. Lett. **B332** (1994) 88

[17] D.R.T. Jones, L. Mezincescu and Y.-P. Yao, Phys. Lett. **B148** (1984) 317

[18] I. Jack and D.R.T Jones, Phys. Lett. **B333** (1994) 372

[19] CDF Collaboration: F. Abe et al., Fermilab preprint, FERMILAB-PUB-94-116-E